SOME OPEN PROBLEMS IN FORMAL CONCEPT ANALYSIS

ABSTRACT. This note intends to collect some open problems in Formal Concept Analysis. The ones below have been presented at the International Conference on Formal Concept Analysis (ICFCA 2006) held in Dresden February 2006. Would you like to add some problems to this list, please feel free to send them to kwuida@math.unibe.ch

PROBLEMS PRESENTED AT ICFCA 2006 IN DRESDEN

Problem 1 (Minimal generators for Boolean layer cakes)**.**

A **Boolean layer cake** is an ordered set obtained from a Boolean lattice $\mathbf{2}^n$ by selecting any number of complete level sets from $\mathbf{2}^n$ and endowing their set union with the order inherited from $\mathbf{2}^n$. Boolean layer cakes are lattices iff, apart from $\{0\}$ and $\{1\}$, a series of consecutive level sets is selected. Finding minimal generating sets for lattices obtained this way is still open. Is there any contextual description of (minimal) generating sets for a given concept lattice?
Related works:

Sc99. *Boolean layer cakes*, Theoret. Comput. Sci. **217** (1999), no.2, 255-278

presented by Jürg Schmid
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Problem 2 (Computing pseudo-intents)**.**

Are the problems:

INSTANCE: A context (G, M, I), a subset $Q \subseteq M$
QUESTION: Is Q a pseudo-intent?

and

INSTANCE: A context (G, M, I), a closed subset $Q \subseteq M$
QUESTION: Is there a pseudo-intent P with $P'' = Q$?

coNP-complete?

Is it possible to compute all pseudo-intents with a (cumulative) polynomial-delay algorithm?
Related works:

KO06. *Counting pseudo-intents and #P-completeness* LNAI **3874** (2006), 306-308
Ku04. S.O. Kuznetsov, *On Complexity of Computing the Duquenne-Guigues Basis*, Journal of Universal Computer Science **10**, (2004), no.8, pp. 927-933.

presented by Sergei O. Kuznetsov
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Problem 3 (Frankl's Problem)**.**

This is a much intriguing problem due to Frankl (around 1979). Translated into the theory of formal concept lattices, the problem becomes: prove or disprove that in any finite formal context, some object belongs to at most half of the concept extends. A survey of partial results together with references is provided by Douglas B. West at `http://www.math.uiuc.edu/ west/openp/unionclos.html`.

Problem 4 (Counting Problems)**.**

In knowledge space theory or in formal concept analysis, many entities are worth being counted with respect to the number of elements or the number of objects/ attributes. In some cases, difficult problems can arise. As an example: for knowledge spaces on n items, what is the largest possible size of a base? Here, a *base* of a knowledge space $(Q, \mathcal{K})$ is the smallest subfamily $\mathcal{B}$ of $\mathcal{K}$ such that any state from $\mathcal{K}$ is the union of some elements from $\mathcal{B}$. In terms of formal concept analysis, one asks for the largest possible number of meet-irreducible concepts in a finite concept lattice when the number of objects equals n. For partial results, in particular solutions for small values of n, see the preprint version of [JV06].

Related works:

FCDT06. Jean-Claude Falmagne, Eric Cosyn, Jean-Paul Doignon and Nicolas Thiéry. *The assessment of knowledge, in theory and practice* LNAI **3874** (2006), 61-79

JV98. R.T. Johnson and T.P. Vaughan, On union-closed families. I, *Journal of Combinatorial Theory, Series A* **84** (1998), 242–249.

presented by Jean-Paul Doignon
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Problem 5 (Automatic generation of concept lattices: combinatoric and isomorphism problem)**.**

Produce all possible concept lattices of a certain order (up to isomorphism).

Related works:

KL98. Kontexte und ihre Begriffe. Adalbert Kerber & Wilfried Lex, march 1998 http://www.mathe2.uni-bayreuth.de/people/kerber.html

EHR02. On the number of distributive lattices. M Erne, J Heitzig, J Reinhold -*the electronic journal of combinatorics*, 2002

HR02. Counting Finite Lattices. J Heitzig, J Reinhold - Algebra Universalis, 2002 - Springer Pages: 43 - 53

presented by Wilfried Lex
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Problem 6 (Size of a clone-free Guigues-Duquenne basis)**.**

Let J be a set of items $\{x_1, \ldots, x_{|J|}\}$, $\mathcal{F}$ a collection of subsets of J and $\varphi_{a,b}$ the mapping

$$\begin{array}{rcll} \varphi_{a,b} & : & 2^J & \rightarrow 2^J \\ X & \mapsto & \varphi_{a,b}(X) := & \begin{cases} (X \setminus \{a\}) \cup \{b\} & \text{if } b \notin X \text{ and } a \in X \\ (X \setminus \{b\}) \cup \{a\} & \text{if } a \notin X \text{ and } b \in X \\ X & \text{elsewhere} \end{cases} \end{array}$$

swapping items a and b. The items a and b are called **clone items** in $\mathcal{F}$ iff for any $F \in \mathcal{F}$, we have $\varphi_{a,b}(F) \in \mathcal{F}$. A collection is said **clone-free** if it does not admit any clone items.

PROBLEM: Does it exist a formal context $\mathbb{K} := (G, M, I)$ such that the collection $\mathcal{P}$ of pseudo-intents of $\mathbb{K}$ is clone-free and the size of $\mathcal{P}$ is exponential in the size of $\mathbb{K}$?

An answer to this question will indicate whether clone items are responsible of the combinatorial explosion of some Guigues-Duquennes basis.

Related works:

GMNR05 Uncovering and reducing hidden combinatorics in Guigues-Duquenne covers. Alain Gély, Raoul Medina, Lhouari Nourine, Yoan Renaud, ICFCA05, Lens

Problem 7 (Semantic of clone items)**.**

Is there any semantic related to clone items? (see also Problem 6).

Explanation: Let (G, M, I) be a formal context such that attributes a and b are not clone. Consider the formal sub-context (G, N, I), with $N \subset M$, such that a and b are clone in (G, N, I). Let $c \in M \setminus N$ such that a and b are no longer clone in $(G, N \cup \{c\}, I)$. In other words, without attribute c in the formal context, attributes a and b have a symmetrical behavior. This symmetrical behavior is lost when considering attribute c in the formal context. Does that make sense for someone? Has this kind of information any value in FCA?

Problem 8 (Essential elements)**.**

Let $\mathbb{K} := (G, M, I)$ be a formal context. A closed set $E \subseteq M$ is said to be **essential closed** in $\mathbb{K}$ iff there exists a pseudo-intent $P \subset M$ such that $E = P''$.

Is the following problem

INSTANCE A formal context (G, M, I) and $E \subseteq M$
QUESTION Is E an essential element of (G, M, I)?

in $\mathcal{NP}$?

Is the following problem

INSTANCE A formal context (G, M, I)
QUESTION The number of essential elements of (G, M, I) ?

$\#\mathcal{P}$-hard?

Other problem: does there exist a formal context (G, M, I) such that the collection $\mathcal{E}$ of essential closed sets has exponential size with respect to (G, M, I)?

Related works:

D91 The core of finite lattices. Vincent Duquenne, Discrete Math 88 (1991)

presented by Raoul Medina
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Problem 9 (Understanding dual bonds)**.**

A **dual Bond** from (G, M, I) to (H, N, J) is a relation $R \subseteq G \times H$ for which it holds that

- for every object $g \in G$, the set g^R is an extent of (H, N, J), and
- for every object $h \in H$, the set h^R is an extent of (G, M, I).

It is easy to see that the set of all such dual bonds is a closure system.
PROBLEM: Give a *natural* small formal context for this closure system.

Related works:

KM06. Markus Krötzsch and Grit Malik. *The Tensor Product as a Lattice of Regular Galois Connections*. LNAI **3874** (2006), 89-104

Problem 10 (Are pseudo-intents difficult to recognize?)**.**

Is the decision problem

Given a formal context (G, M, I) and a set $X \subseteq M$.
Is X a pseudo-intent?

in $\mathcal{NP}$?

This is an old problem. It has been proposed e.g. to the Dresden conference on Discrete Mathematics in 2002. See also Problem 2

Problem 11 (Closure reflecting families)**.**

Let $X \mapsto X''$ be a closure operator on M. For a subset $R \subseteq M$ define

$$X^{RR} := (X \cap R)'' \cap R.$$

A family $\mathcal{R} \subseteq \mathfrak{P}(M)$ of subsets of M is called **closure reflecting** if

$$X = X'' \text{ iff } X = X^{RR} \text{ for all } R \in \mathcal{R}.$$

PROBLEM: Find small closure reflecting sets.

Problem 12 (Are formal contexts small on average?)**.**
For a finite lattice L, let

$$r(L) := \frac{|L|}{|J(L)| \cdot |M(L)|},$$

and let

$$\bar{r}(n) := \text{Average}\{r(L) \mid |L| = n\}.$$

PROBLEM: Is $r(n) \in O(n^k)$ for some k?

presented by Bernhard Ganter
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Problem 13 (Handling large contexts)**.**

How to calculate/generate all concepts of a large context (e.g. 120 000 objects × 70 000 attributes)?
Application: formal contexts that are sparse matrices (i.e. have a relatively small number of crosses per row and column)

Input: the formal context in form of a comma-delimited list of object/attribute pairs.
Desired output: the following lists (in comma-delimited format)

- concept/concept relation (list representation of Hasse diagram)
- concept/extent relation
- concept/intent relation
- concept/contingent objects relation
- concept/contingent attributes relation

This should be an open-source software tool.

Related works:

some algorithms in the data mining community.

presented by John Old & Uta Priss
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Problem 14 (Tree-order).

A finite poset $(P, \leq)$ is a **directed tree-order** if it has a greatest element and for all $x \in P$, $\uparrow x$ is a chain. Each finite poset can be decomposed as unions of tree-orders. Let k be the minimal number. What is the mean of k? What is the relation to the order dimension of $(P, \leq)$?

presented by Sandor Radelezcki
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Problem 15 (Systems of pseudo-intents in a fuzzy setting).

Let $\mathbf{L}$ be a complete residuated lattice (structure of truth degrees), let $\langle X, Y, I\rangle$ be a formal fuzzy context, i.e. I maps $X \times Y$ to the support set of $\mathbf{L}$. Under which conditions a system of pseudo-intents of $\langle X, Y, I\rangle$ exists? How systems of pseudointents can be computed?
Related works:

BV06. Radim Bělohlávek and Vilém Vychodil. *Attribute implications in a fuzzy setting*. LNAI **3874** (2006), 45-60

presented by Radim Bělohlávek and Vilém Vychodil
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Problem 16 (Peeling).

This problem involves peeling, a decomposition technique used on "massive" graphs. When a formal context is viewed as a bigraph, the peeling technique yields a partition of the union of the object and attribute sets of the context, and obviously a partition of each set, when localized via intersection. The problem is to relate this partition to the concept lattice of a formal context.

The original context is called the (-1)-core, and for each n, n-peeling iteratively deletes objects and attributes of at most degree n from the (n-1)-core, until no more deletion is possible. An object or attribute is in the n-core if it is deleted during n-peeling. The core is the M-core provided that no objects or attributes remain after M-peeling. For example, if a context has non-empty g' and m', for all $g \in G$ and $m \in M$, then the (-1)-core equals the 0-core (no objects or attributes are removed when 0-peeling is applied). Another obvious fact is that if the bipartite graph associated with a formal context is a tree, then the 1-core is the core.

QUESTIONS: How does this decomposition method for contexts translate into a decomposition of the corresponding concept lattices?
Are there any order- or lattice-theoretic properties that are naturally related to the peeling decomposition?
Are there relationships between the peeling decomposition of a context and restrictions on sets of implications or associations rules (bounded in some way) derived from a context?

Related works:

A more technical definition appeared in a forthcoming paper with J. Abello, in an AMS-DIMACS Special Volume (book) on Data Mining and Epidemiology.

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